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NONLINEAR ANTIPLANE DEFORMATION OF AN ELASTIC BODY

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The antiplane elastic deformation of a homogeneous isotropic prestretched cylindrical body is studied in a nonlinear formulation in actual-state variables under incompressibility conditions, the absence of volume forces, and under constant lateral loading along the generatrix. The boundary-value problem of axial displacement is obtained in Cartesian and complex variables and sufficient ellipticity conditions for this problem are indicated in terms of the elastic potential. The similarity to a plane vortex-free gas flow is established. The problem is solved for Mooney and Rivlin–Sonders materials simulating strong elastic deformations of rubber-like materials. Axisymmetric solutions are considered.

Many important results of the theory of elasticity have been obtained in studying particular types of deformation. In these cases, the solution of the problems is simplified: the number of equations, unknown functions, and their arguments is decreased. In a number of cases, a complex analysis, which simplifies appreciably the treatment, is applicable to similar deformations. Along with plane and axisymmetrical deformations, antiplane deformation is also a particular type of deformation. Various aspects of antiplane deformation were studied in [1–4]; below, this deformation is considered in a nonlinear formulation in actual-state variables as applied to a cylindrical body.

Upon antiplane deformation of a cylindrical body, the displacements of its particles are parallel to the generatrix and they do not depend on the axial coordinate. In the actual-state Cartesian coordinate system x_1, x_2, x_3 with the x_3 axis parallel to the cylinder generatrix and the $x_1 = x$ and $x_2 = y$ axes in the plane of its average transverse cross section, in the presence of a preliminary uniform volume-preserving tension this deformation is determined by the displacements

$$u_1 = (1 - e)x_1, \quad u_2 = (1 - e)x_2, \quad u_3 = (1 - e^{-2})x_3 + w(x_1, x_2), \tag{1}$$

where e is the specified constant (for $e = 1$, preliminary tension is absent) and $w(x, y)$ is a double continuously differentiable in the transverse cross section S (with boundary L) function to be determined. We obtain static relations of nonlinear elasticity in this case, assuming that the material is homogeneous and isotropic with a given internal energy, the volume forces are absent, and the external surface forces do not depend on the axial coordinate.

In actual-state variables, the strain and stress measures are the symmetrical Almansi and Cauchy tensors E_{kl} and Cauchy P_{kl}, respectively, and Murnaghan's law, which relates these tensors, is a constitutive equation [5]. In antiplane deformation (1), the strain components E_{kl} and the basis strain invariants E_k determined in the form

$$2E_{kl} = \frac{\partial u_l}{\partial u_k} + \frac{\partial u_k}{\partial u_l} - \frac{\partial u_m}{\partial x_k}\frac{\partial u_m}{\partial x_l}, \quad E_1 = E_{mm}, \quad 2E_2 = E_{nn}E_{mm} - E_{nm}E_{mn}, \quad E_3 = |E_{kl}|$$

(summation is performed over repeated indices) are expressed in terms of the axial displacement $w(x, y)$

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$$E_{11} = \frac{1-e^2}{2} - \frac{1}{2}\Big(\frac{\partial w}{\partial x}\Big)^2, \quad E_{22} = \frac{1-e^2}{2} - \frac{1}{2}\Big(\frac{\partial w}{\partial y}\Big)^2, \quad E_{33} = \frac{1-e^{-4}}{2},$$
$$E_{12} = -\frac{1}{2}\,\frac{\partial w}{\partial x}\,\frac{\partial w}{\partial y}, \quad E_{23} = \frac{e^{-2}}{2}\,\frac{\partial w}{\partial y}, \quad E_{31} = \frac{e^{-2}}{2}\,\frac{\partial w}{\partial x}; \tag{2}$$

$$2E_1 = -(e-e^{-1})^2(2+e^{-2}) - |\nabla w|^2, \quad 4E_2 = (e-e^{-1})^2(e^2-2-2e^{-2}) - (2-e^2)|\nabla w|^2,$$
$$8E_3 = (e-e^{-1})^2(e^2-e^{-2}) - (1-e^2)|\nabla w|^2 \tag{3}$$

and, hence, they are functions of the transverse coordinates of the cylinder.

For invariants (3), the following relations are true:

$$E_1 \leqslant 0, \qquad 4E_2 = (e-e^{-1})^2(1-e^2) + 2(2-e^2)E_1,$$
$$8E_3 = (e-e^{-1})^2(1-e^2) + 2(1-e^2)E_1, \qquad E_1 - 2E_2 + 4E_3 = 0. \tag{4}$$

By virtue of the above relations, the continuity equation takes the form of the incompressibility condition

$$\rho = \rho_0\sqrt{1-2E_1+4E_2-8E_3} = \rho_0,$$

where ρ_0 and ρ are the initial and actual-state densities of the material, respectively. Therefore, the material is incompressible in antiplane deformation.

For an incompressible material, Murnaghan's law has the form [5, 6]

$$P_{kl} = -q_0\delta_{kl} + (\delta_{km} - 2E_{km})\,\frac{\partial U}{\partial E_{lm}}, \tag{5}$$

where U is the internal-energy density (elastic potential) and q_0 is the Lagrangian factor. For a homogeneous isotropic material, the elastic potential is a function of basis strain invariants. By virtue of relations (4) between the invariants, this potential depends only on the first invariant: $U = U(E_1)$.

Using the expression for the tensor gradient of the first invariant [6] $\partial E_1/\partial E_{lm} = \delta_{ml}$, we obtain $\partial U/\partial E_{lm} = U'\delta_{ml}$. Then, it follows from (5) that upon antiplane deformation, Murnaghan's law can be presented by a quasilinear dependence of stresses on strains:

$$P_{kl} = -q\delta_{kl} - 2U'(E_1)E_{kl} \qquad (q = q_0 - U'). \tag{6}$$

Here q is the hydrostatic pressure. The consequence of formulas (2) and (6) is the expression for stresses in terms of the hydrostatic pressure and axial displacement:

$$P_{11} = -q - (1-e^2)U' + U'\Big(\frac{\partial w}{\partial x}\Big)^2, \qquad P_{22} = -q - (1-e^2)U' + U'\Big(\frac{\partial w}{\partial y}\Big)^2,$$
$$P_{33} = -q - (1-e^{-4})U', \quad P_{12} = U'\,\frac{\partial w}{\partial x}\,\frac{\partial w}{\partial y}, \quad P_{23} = -e^{-2}U'\,\frac{\partial w}{\partial y}, \quad P_{31} = -e^{-2}U'\,\frac{\partial w}{\partial x}. \tag{7}$$

The external normals on the upper (S^+) and lower (S^-) bases of the cylinder have the constant components $(n_l^{\pm}) = (0, 0, \pm 1)$; on its lateral surface S^*, these components depend on transverse coordinates: $(n_l) = (n_1(x,y), n_2(x,y), 0)$. Therefore, depending on the displacement and pressure, on the corresponding sites, the stress vectors $(p_k^{\pm})$ and (p_k) presented by formulas $p_k^{\pm} = P_{kl}n_l^{\pm} = \pm P_{k3}$ and $p_k = P_{kl}n_l = P_{k1}n_1 + P_{k2}n_2$ have the form

$$p_1^{\pm} = \mp e^{-2}U'\,\frac{\partial w}{\partial x}, \quad p_2^{\pm} = \mp e^{-2}U'\,\frac{\partial w}{\partial y}, \quad p_3^{\pm} = \mp[q + (1-e^{-4})U']; \tag{8}$$

$$p_1 = -[q + (1-e^2U')]n_1 + U'\,\frac{\partial w}{\partial n}\,\frac{\partial w}{\partial x}, \quad p_2 = -[q + (1-e^2U')]n_2 + U'\,\frac{\partial w}{\partial n}\,\frac{\partial w}{\partial y},$$
$$p_3 = -e^{-2}U'\,\frac{\partial w}{\partial n}. \tag{9}$$

It follows from (8) and (9) that if the surface load does not depend on x_3, then x_3 does not depend on q on the surface as well. In this case, one can assume that the pressure does not depend on this coordinate in the entire volume of the cylinder: $q = q(x, y)$. Then, according to (7), all the stresses do not depend on x_3 as well: $P_{kl} = P_{kl}(x, y)$. By virtue of this property and the representations (7) for stresses, in the absence of volume forces ($\partial P_{kl}/\partial x_l = 0$) the equation of equilibrium are equations for determination of the pressure and displacement in the cross section S of the cylinder:

$$\frac{\partial}{\partial x}\left[-q-(1-e^2)U'+U'\Big(\frac{\partial w}{\partial x}\Big)^2\right]+\frac{\partial}{\partial y}\Big(U'\,\frac{\partial w}{\partial x}\,\frac{\partial w}{\partial y}\Big)=0,$$

$$\frac{\partial}{\partial y}\left[-q-(1-e^2)U'+U'\Big(\frac{\partial w}{\partial y}\Big)^2\right]+\frac{\partial}{\partial x}\Big(U'\,\frac{\partial w}{\partial x}\,\frac{\partial w}{\partial y}\Big)=0, \tag{10}$$

$$\frac{\partial}{\partial x}\Big(U'\,\frac{\partial w}{\partial x}\Big)+\frac{\partial}{\partial y}\Big(U'\,\frac{\partial w}{\partial y}\Big)=0.$$

In system (10), the first two equations can be transformed by means of the third one to the form

$$-\frac{\partial[q+(1-e^2)U']}{\partial x}+U'\,\frac{\partial}{\partial x}\,\frac{|\nabla w|^2}{2}=0, \qquad -\frac{\partial[q+(1-e^2)U']}{\partial y}+U'\,\frac{\partial}{\partial y}\,\frac{|\nabla w|^2}{2}=0.$$

With allowance for the relations

$$\frac{\partial E_1}{\partial x_n}=-\frac{\partial}{\partial x_n}\,\frac{|\nabla w|^2}{2}, \qquad U'\,\frac{\partial}{\partial x_n}\,\frac{|\nabla w|^2}{2}=-U'\,\frac{\partial E_1}{\partial x_n}=-\frac{\partial U}{\partial x_n} \qquad (n=1,\,2),$$

these equations are written in the form

$$\frac{\partial}{\partial x}\,[q+(1-e^2)U'+U]=0, \qquad \frac{\partial}{\partial y}\,[q+(1-e^2)U'+U]=0$$

and give, after integration, a representation of hydrostatic pressure via the elastic potential

$$q=h-(1-e^2)U'-U, \tag{11}$$

where h is an integration constant. At the cylinder butt-ends, according to (8) and (11), the axial forces $Q_3^{\pm}$ depend on the constant h:

$$Q_3^{\pm}=\int\limits_S p_3^{\pm}\,dS=\mp\left[hS-\int\limits_S[U+(e^{-4}-e^2)U']\,dS\right];$$

therefore, h can be determined from the condition of the absence of axial forces:

$$h=\frac{1}{S}\int\limits_S[U+(e^{-4}-e^2)U']\,dS \tag{12}$$

(for $e = 1$, h is the average value of the potential U in the region S and q is the deviation of the potential from its average value).

Written with allowance for the dependences $U'(E_1)$ and $E_1(\nabla w)$, in extended form, together with the geometrical condition at the boundary of the region, the third equation in (10) constitutes the nonlinear boundary-value problem of axial displacement

$$\left[U'-U''\Big(\frac{\partial w}{\partial x}\Big)^2\right]\frac{\partial^2 w}{\partial x^2}-2U''\frac{\partial w}{\partial x}\,\frac{\partial w}{\partial y}\,\frac{\partial^2 w}{\partial x\,\partial y}+\left[U'-U''\Big(\frac{\partial w}{\partial y}\Big)^2\right]\frac{\partial^2 w}{\partial y^2}=0, \quad w\Big|_L=w^*. \tag{13}$$

Determined from the solution of this problem, the axial displacement allows one to find all the desired quantities: displacements (1), hydrostatic pressure (11), stresses (7), and the permissible surface loads (8) and (9).

To establish the type of Eq. (13), following [7], we consider the characteristic determinant D. In this case, the determinant is a quadratic polynomial in the variables r and s and it can be presented in the form

$$D = \left[U' - U''\Big(\frac{\partial w}{\partial x}\Big)^2\right] r^2 - 2U'' \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} rs + \left[U' - U''\Big(\frac{\partial w}{\partial y}\Big)^2\right] s^2 = U'(r^2+s^2) - U''\Big(r\frac{\partial w}{\partial x} + s\frac{\partial w}{\partial y}\Big)^2.$$

One can conclude that

$$D > 0 \quad \text{for} \quad U' > 0, \quad U'' \leqslant 0; \qquad D < 0 \quad \text{for} \quad U' < 0, \quad U'' \geqslant 0. \tag{14}$$

If conditions (14) are satisfied, the characteristic equation $D = 0$ has no real roots; therefore, Eq. (13) is an elliptic-type equation for any solution. Thus, inequalities (14) are the sufficient conditions for ellipticity of the equations of antiplane deformation of an incompressible elastic material presented in terms of the elastic potential. It is shown in [8] that Eqs. (14) are also the ellipticity conditions upon plane deformation of an incompressible material.

One can establish the analogy between the nonlinear antiplane deformation of an elastic material and a steady-state plane vortex-free gas flow. Indeed, introducing the strength characteristic of the material (coefficient of elasticity) related to the elastic potential and the ellipticity condition $k^2 = -U'/U''$, one can write Eq. (13) in the form

$$\left[1 + \frac{1}{k^2}\Big(\frac{\partial w}{\partial x}\Big)^2\right]\frac{\partial^2 w}{\partial x^2} + \frac{2}{k^2}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\,\partial y} + \left[1 + \frac{1}{k^2}\Big(\frac{\partial w}{\partial y}\Big)^2\right]\frac{\partial^2 w}{\partial y^2} = 0 \tag{15}$$

and compare it with the equation for the velocity potential φ in plane vortex-free flow of an ideal gas with the local velocity of sound a^2 [9]:

$$\left[1 - \frac{1}{a^2}\Big(\frac{\partial \varphi}{\partial x}\Big)^2\right]\frac{\partial^2 \varphi}{\partial x^2} - \frac{2}{a^2}\frac{\partial \varphi}{\partial x}\frac{\partial \varphi}{\partial y}\frac{\partial^2 \varphi}{\partial x\,\partial y} + \left[1 - \frac{1}{a^2}\Big(\frac{\partial \varphi}{\partial y}\Big)^2\right]\frac{\partial^2 \varphi}{\partial y^2} = 0. \tag{16}$$

Equation (15) is similar to Eq. (16) [the quantities φ and $-a^2$ in (16) correspond to the quantities w and k^2 in (15)]. This analogy allows us to apply the methods of gas dynamics to the problems of nonlinear elasticity.

With a certain accuracy, strong elastic deformations of incompressible rubber-like materials can be described, for example, by the Mooney U_1 or generalizing Rivlin–Sonders U_2 potential [2]:

$$U_1 = C_1(I_1 - 3) + C_2(I_2 - 3), \qquad U_2 = C_1(I_1 - 3) + f(I_2 - 3). \tag{17}$$

Here $C_1 > 0$ and $C_2 \geqslant 0$ are elastic constants, I_1 and I_2 are special strain invariants, and f is a positive function. Under incompressibility conditions, the invariants I_1 and I_2 are expressed in terms of the invariants E_1 and E_2 of the Almansi tensor and, by virtue of (4), they are presented in terms of the invariant E_1:

$$I_1 - 3 = 4(E_2 - E_1) = (1 - e^2)(e - e^{-1})^2 - 2e^2 E_1, \qquad I_2 - 3 = -2E_1.$$

For potentials (17), these relations make it possible to obtain the representations

$$U_1 = C_1(1 - e^2)(e - e^{-1})^2 - 2(e^2 C_1 + C_2)E_1, \quad U_2 = C_1(1 - e^2)(e - e^{-1})^2 - 2e^2 C_1 E_1 + f(-2E_1). \tag{18}$$

With allowance for the inequalities $E_1 \leqslant 0$, it follows from expressions (18) that, for the positiveness of the potentials, it is necessary that $e^2 \leqslant 1$.

For the Mooney potential in (18), the ellipticity conditions (14) is satisfied: $U' = -2(e^2 C_1 + C_2) < 0$ and $U'' = 0$, and problem (13) is the Dirichlet problem for axial displacement (coinciding with a similar problem in the theory of linear elasticity):

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0, \qquad w\Big|_L = w^*. \tag{19}$$

In this case, the same displacement fields and different stress fields correspond to antiplane deformation in the linear and nonlinear theories.

When f is a linear function, the Rivlin–Sonders potential coincides with the Mooney potential considered above. We consider the case where f is a quadratic function: $f(-2E_1) = l(-2E_1)^2 + m(-2E_1) + n$. Then, the elastic potential is also quadratic:

$$U = aE_1^2 - 2bE_1 + c, \quad a = 4l, \quad b = m + e^2 C_1, \quad c = n + C_1(1 - e^2)(e - e^{-1})^2. \tag{20}$$

For the potential (20), the ellipticity conditions (14) are also satisfied: $U'=-a(|\nabla w|^2+v^2)<0$, $U''=2a>0$ $[v^2=(e-e^{-1})^2(2+e^{-2})+2b/a]$, and the displacement problem (13) takes the form

$$\left[v^2+3\Big(\frac{\partial w}{\partial x}\Big)^2+\Big(\frac{\partial w}{\partial y}\Big)^2\right]\frac{\partial^2 w}{\partial x^2}+4\,\frac{\partial w}{\partial x}\,\frac{\partial w}{\partial y}\,\frac{\partial^2 w}{\partial x\,\partial y}+\left[v^2+\Big(\frac{\partial w}{\partial x}\Big)^2+3\Big(\frac{\partial w}{\partial y}\Big)^2\right]\frac{\partial^2 w}{\partial y^2}=0,$$
$$w\Big|_L=w^*. \tag{21}$$

The boundary-value problems of axial displacement can be also formulated in the complex variables $z=x+iy$ and $\bar z=x-iy$, in which the displacement becomes a function of the form $w=w(z,\bar z)$. For the Mooney material, in complex variables, problem (19) has the form

$$\frac{\partial^2 w}{\partial z\,\partial\bar z}=0,\qquad w\Big|_L=w^*. \tag{22}$$

The general solution of the harmonic equation is expressed in terms of the analytical function $\varphi(z)$, which is determined relative to the specified real part at the boundary:

$$2w=\varphi(z)+\bar\varphi(\bar z),\qquad \operatorname{Re}\varphi(z)\Big|_L=w^*. \tag{23}$$

In complex variables, for a Rivlin–Sonders material (with a quadratic elastic potential), problem (21) takes the form

$$\Big(\frac{v^2}{2}+4\,\frac{\partial w}{\partial z}\,\frac{\partial w}{\partial\bar z}\Big)\frac{\partial^2 w}{\partial z\,\partial\bar z}+\Big(\frac{\partial w}{\partial\bar z}\Big)^2\frac{\partial^2 w}{\partial z^2}+\Big(\frac{\partial w}{\partial z}\Big)^2\frac{\partial^2 w}{\partial\bar z^2}=0,\qquad w\Big|_L=w^*. \tag{24}$$

With the parameter v^{-2} smaller than unity, one can find an approximate solution by representing the displacement as a series $w=\sum\limits_{k=0}^{\infty}w_k v^{-2k}$ and keeping a finite number of terms in it. Substituting the series into the equation and the boundary condition (24) and equating the coefficients at equal powers of the parameter in different parts, we obtain a sequence of linear problems for the displacement components w_k:

$$\frac{\partial^2 w_k}{\partial z\,\partial\bar z}+N_{k-1}(z,\bar z)=0,\qquad w_k\Big|_L=w^*\delta_{0k}\qquad (k=0,1,2,\ldots). \tag{25}$$

Here N_{k-1} are known functions determined by the previous approximations

$$N_{k-1}=2\sum_{m=0}^{k-1}\Big(4G_m\,\frac{\partial^2 w_{k-1-m}}{\partial z\,\partial\bar z}+\bar H_m\,\frac{\partial^2 w_{k-1-m}}{\partial z^2}+H_m\,\frac{\partial^2 w_{k-1-m}}{\partial\bar z^2}\Big),$$

$$G_m=\sum_{n=0}^{m}\frac{\partial w_n}{\partial z}\,\frac{\partial w_{m-n}}{\partial\bar z},\qquad H_m=\sum_{n=0}^{m}\frac{\partial w_n}{\partial z}\,\frac{\partial w_{m-n}}{\partial z}.$$

In particular, for the zeroth approximation $w=w_0$, problem (25) coincides with problem (22) and has the solution (23):

$$\frac{\partial^2 w_0}{\partial z\,\partial\bar z}=0,\quad w_0\Big|_L=w^*,\quad 2w_0=\varphi_0(z)+\bar\varphi_0(\bar z),\quad \operatorname{Re}\varphi_0(z)\Big|_L=w^*.$$

For the displacement w_1 determining the first approximation $w=w_0+v^{-2}w_1$, this problem takes the form

$$4\,\frac{\partial^2 w_1}{\partial z\,\partial\bar z}+(\bar\varphi_0'(\bar z))^2\varphi_0''(z)+(\varphi_0'(z))^2\bar\varphi_0''(\bar z)=0,\qquad w_1\Big|_L=0$$

and has the solution

$$4w_1+\varphi_0'(z)\int(\bar\varphi_0'(\bar z))^2\,d\bar z+\bar\varphi_0'(\bar z)\int(\varphi_0'(z))^2\,dz=\varphi_1(z)+\bar\varphi_1(\bar z),$$

$$\operatorname{Re}\varphi_1(z)\Big|_L=\operatorname{Re}\left(\bar\varphi_0'(z)\int(\varphi_0'(z))^2\,dz\right)\Big|_L.$$

We consider the antiplane deformation of a tube made from an incompressible (Mooney or Rivlin–Sonders) material and pressed in coaxial rigid cylindrical cartridges of radii r_1 and r_2 ($r_1 > r_2$) for the specified displacement of an external cartridge, for a fixed internal cartridge, and in the absence of preliminary deformation:

$$w\Big|_{r=r_1} = w_1, \qquad w\Big|_{r=r_2} = 0, \qquad e = 1. \tag{26}$$

For a Mooney material (18) whose elastic potential and its derivative for $e = 1$ have the values

$$U = -2CE_1 = C\Big|\nabla w\Big|^2, \qquad U' = -2C, \qquad C = C_1 + C_2, \tag{27}$$

the axial displacement is determined from problem (19). Assuming that the displacement is symmetrical $w = w(r)$ (r is the polar radius), we write the harmonic equation in the form $(4r)^{-1}(rw')' = 0$. The general solution $w = A \ln r + B$ contains arbitrary constants A and B. Determining them from conditions (26), we obtain a solution of the problem in the form

$$w = A \ln(r/r_2), \qquad A = w_1/\ln(r_1/r_2), \qquad r = \sqrt{x^2 + y^2}. \tag{28}$$

Using (12), (27), and (28), we find

$$|\nabla w| = \frac{A}{r}, \qquad U = C\,\frac{A^2}{r^2}, \qquad h = \frac{1}{S}\int_S U\,dS = CA\,\frac{2w_1}{r_1^2 - r_2^2};$$

therefore, the hydrostatic pressure (11) is equal to

$$q = h - U = CA\Big(\frac{2w_1}{r_1^2 - r_2^2} - \frac{A}{r^2}\Big). \tag{29}$$

Thus, the displacement and pressure in the tube depend only on the polar radius.

The Cartesian stresses (7) in the tube volume $P_{11} = -q - 2CA^2x^2/r^4$, $P_{22} = -q - CA^2y^2/r^4$, $P_{33} = -q$, $P_{12} = -2CA^2xy/r^4$, $P_{23} = 2CAy/r^2$, and $P_{31} = 2CAx/r^2$ correspond to the quantities in (28) and (29). According to (28) and (29), with allowance for the representations of the components of the normal and the normal derivative of displacement $n_1 = x/r$ and $n_2 = y/r$ and $\partial w/\partial n = A/r$, the Cartesian stresses (9) on the cylindrical surfaces of the tube are determined in the form

$$p_1\Big|_{r_i} = \mp CA\Big(\frac{2w_1}{r_1^2 - r_2^2} + \frac{A}{r_i^2}\Big)\frac{x_i}{r_i}, \quad p_2\Big|_{r_i} = \mp CA\Big(\frac{2w_1}{r_1^2 - r_2^2} + \frac{A}{r_i^2}\Big)\frac{y_i}{r_i}, \quad p_3\Big|_{r_i} = \pm CA\,\frac{2}{r_i}. \tag{30}$$

Hereinafter, $i = 1$ and 2 ($i = 1$ and the upper sign corresponds to the external cylinder, and $i = 2$ and the lower sign corresponds to the internal cylinder). Accordingly, referred to the unit length in the longitudinal direction of the tube, the components $F_k\Big|_{r_i}$ of the main vectors of lateral surface forces have the zero transverse and nonzero longitudinal components:

$$F_1\Big|_{r_i} = \iint p_1\Big|_{r_i} r_i\,d\varphi\,dx_3 = 0, \quad F_2\Big|_{r_i} = \iint p_2\Big|_{r_i} r_i\,d\varphi\,dx_3 = 0, \quad F_3\Big|_{r_i} = \pm 4\pi CA \quad (i = 1, 2)$$

(φ is the polar angle).

According to (8), the surface stresses at the butt-ends of the tube have the values

$$p_1^{\pm} = \pm 2CA\,\frac{x}{r^2}, \qquad p_2^{\pm} = 2CA\,\frac{y}{r^2}, \qquad p_3^{\pm} = \pm CA\Big(\frac{A}{r^2} - \frac{2w_1}{r_1^2 - r_2^2}\Big), \tag{31}$$

where the upper and lower signs correspond to the upper and lower butt-ends. The components $Q_k^{\pm}$ of the main vectors of these forces are zero: $Q_k^{\pm} = \iint p_k^{\pm} r\,d\varphi\,dr = 0$.

In the axisymmetric problem considered, in the cylindrical coordinate system r, φ, z, similarly to the displacement and pressure, the physical tensor and stress-vector components (determined in terms of the Cartesian components of corresponding quantities from the transformation formulas) depend only on the polar radius; here the stresses at the lateral surface of the tube are constant quantities:

$$P_{rr} = -CA\Big(\frac{A}{r^2} + \frac{2w_1}{r_1^2 - r_2^2}\Big), \quad P_{\varphi\varphi} = P_{zz} = CA\Big(\frac{A}{r^2} - \frac{2w_1}{r_1^2 - r_2^2}\Big), \quad P_{r\varphi} = P_{\varphi z} = 0,$$

$$P_{zr} = CA\,\frac{2}{r}, \quad p_r^{\pm} = \pm CA\,\frac{2}{r}, \quad p_\varphi^{\pm} = 0, \quad p_z^{\pm} = \pm CA\Big(\frac{A}{r^2} - \frac{2w_1}{r_1^2 - r_2^2}\Big),$$

$$p_r\Big|_{r_i} = -CA\Big(\frac{A}{r_i^2} + \frac{2w_1}{r_1^2 - r_2^2}\Big), \qquad p_\varphi\Big|_{r_i} = 0, \qquad p_z\Big|_{r_i} = CA\,\frac{2}{r_i}.$$

Upon deformation of a tube made from a Rivlin–Sonders material, the displacement should be determined from Eq. (21), which takes the form $(8r)^{-1}(rw'^3 + v^2 rw')' = 0$ for $w = w(r)$. After integration, for the derivative w', we obtain the incomplete cubic equation $w'^3 + v^2 w' - m/r = 0$ (m is an arbitrary constant), which has only one real solution owing to the inequality $(v^2/3)^3 + (-m/(2r))^2 > 0$ [10]:

$$w' = J_+(r,m) + J_-(r,m), \qquad J_\pm(r,m) = \sqrt[3]{\frac{m}{2r} \pm \sqrt{\frac{v^6}{27} + \frac{m^2}{4r^2}}}. \tag{32}$$

Integration of Eq. (32) under conditions (26) determines the displacement and yields the following relation for finding the constant m:

$$w = \int_{r_2}^{r} (J_+(r,m) + J_-(r,m))\,dr, \qquad w_1 = \int_{r_2}^{r_1} (J_+(r,m) + J_-(r,m))\,dr. \tag{33}$$

The integrals in (33) admit the representation in terms of elementary functions: the integrand is reduced to a rational function by means of the substitution $t = 3J_+^2(r)$ $[r = m(\sqrt{3t}\,)^3/(t^3 - v^6)]$, and, according to [11], the integrals are taken in the finite form:

$$w = m\left[\frac{3t(v^2 - t)}{v^6 - t^3} - \frac{1}{2v^2}\ln\frac{(v^2 - t)^2}{v^4 + v^2 t + t^2}\right]_{t_2}^{t}, \quad w_1 = m\left[\frac{3t(v^2 - t)}{v^6 - t^3} - \frac{1}{2v^2}\ln\frac{(v^2 - t)^2}{v^4 + v^2 t + t^2}\right]_{t_2}^{t_1},$$

where $t_1 = 3J_+^2(r_1)$ and $t_2 = 3J_+^2(r_2)$.

In (20), for U and $e = 1$, we assume $c = 0$, which corresponds to vanishing of the elastic potential in the absence of deformation. In the case considered, the elastic potential and related quantities are expressed in terms of the derivative $w'(r)$ determined by formula (32); therefore, they are functions of the polar radius:

$$U = \frac{a}{4}\,w'^2(r)(w'^2(r) + 2v^2), \qquad U' = a(w'^2(r) + v^2),$$
$$q = h - \frac{a}{4}\,w'^2(r)(w'^2(r) + 2v^2), \quad h = \frac{a}{2(r_1^2 - r_2^2)}\int_{r_2}^{r_1} w'^2(r)(w'^2(r) + 2v^2)\,r\,dr, \quad v^2 = \frac{2b}{a}. \tag{34}$$

The functions of the polar radius are also the cylindrical stress-tensor components in the tube volume and the stress-vector components on its surface:

$$P_{rr} = -\frac{a}{4}\,w'^2(r)(3w'^2(r) + 2v^2) - h, \quad P_{\varphi\varphi} = P_{zz} = \frac{a}{4}\,w'^2(r)(w'^2(r) + 2v^2) - h,$$
$$P_{r\varphi} = P_{\varphi z} = 0, \quad P_{zr} = aw'(r)(w'^2(r) + v^2), \quad p_r^{\pm} = \pm aw'(r)(w'^2(r) + v^2), \quad p_\varphi^{\pm} = 0,$$
$$p_z^{\pm} = \pm\Big[\frac{a}{4}\,w'^2(r)(w'^2(r) + 2v^2) - h\Big], \quad p_r\Big|_{r_i} = \mp\Big[h + \frac{a}{4}\,w'^2(r_i)(3w'^2(r_i) + 2v^2)\Big],$$
$$p_\varphi\Big|_{r_i} = 0, \qquad p_z\Big|_{r_i} = \pm aw'(r_i)(w'^2(r_i) + v^2). \tag{35}$$

If $v^{-2} \ll 1$, one can obtain approximate expressions for the displacement and its derivative, and, hence, for the solution (32)–(35) of the axisymmetric problem. In the case of the linearly elastic potential

w', the equation for $rw' = A$ is an approximation of the corresponding equation for the quadratic potential $rw'^3 + v^2 rw' = m$ if $m = Av^2$. Then, presenting the radicals in (32) by means of the expansions

$$J_+ = \frac{v}{\sqrt{3}}\left(1 + \frac{\sqrt{3}A}{2r}\,v^{-1} + \frac{3A^2}{8r^2}\,v^{-2} - \frac{\sqrt{3}A^3}{2r^3}\,v^{-3} + \frac{135A^4}{128r^4}\,v^{-4}\right),$$

$$J_- = -\frac{v}{\sqrt{3}}\left(1 - \frac{\sqrt{3}A}{2r}\,v^{-1} + \frac{3A^2}{8r^2}\,v^{-2} + \frac{\sqrt{3}A^3}{2r^3}\,v^{-3} + \frac{135A^4}{128r^4}\,v^{-4}\right),$$

in the indicated approximation, we obtain the expressions

$$w = A\ln r + \frac{A^3}{2v^2r^2} + B, \qquad w' = \frac{A}{r}\Big(1 - \frac{A^2}{v^2r^2}\Big) \qquad (A,\, B = \text{const})$$

for the displacement and its derivative, respectively. With allowance for the boundary conditions (26), the displacements is equal to

$$w = A\ln\frac{r}{r_2} + \frac{A^3}{2v^2}\Big(\frac{1}{r^2} - \frac{1}{r_2^2}\Big), \tag{36}$$

where the constant A is determined from the cubic equation

$$A^3 - ugA + uw_1 = 0 \qquad \Big(u = \frac{2r_1^2r_2^2v^2}{r_1^2 - r_2^2}, \qquad g = \ln\frac{r_1}{r_2}\Big).$$

The equation has a unique real solution under the condition from [10]:

$$\Big(-\frac{ug}{3}\Big)^3 + \Big(\frac{uw_1}{2}\Big)^2 > 0 \qquad \left[\frac{w_1^2}{4} - \frac{2}{27}\,\frac{r_1^2r_2^2v^2}{r_1^2 - r_2^2}\Big(\ln\frac{r_1}{r_2}\Big)^3 > 0\right].$$

The approximate nonlinear solution (36) generalizes the linear solution (28).

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